Workshop 8

Experiment design guided by estimates of uncertainty

This workshop is about performing an uncertainty analysis before sensors have been selected, before an apparatus has been built, and before you have any data. This analysis is done in the *design phase* of an experimental program.

The purpose of this type of analysis is to inform your design decisions based on evidence that realistic bounds on the *expected uncertainty in the result* can be met. When the analysis is complete, you will have:

1. determined the feasibility of a proposed design before prototyping and shaking-down,
2. determined design features of the apparatus,
3. selected instrumentation with systematic uncertainties appropriate to each measurand, and
4. developed an uncertainty budget for design.

## Please learn to define the following terms.

|  |  |
| --- | --- |
| difference between a design range and a design set-point |  |
| relative uncertainty |  |
| RSSC |  |
| UMF |  |
| UPC |  |
| uncertainty budget |  |

## Instructions

We begin with a lecture to introduce the topics. Afterwards, you apply these concepts in an extended design example. Individual work is required, but obtaining the help of your neighbors is encouraged.

* Consider doing the calculations in Excel or MATLAB to make it easy to correct errors as well as play “what-if” with your analysis.
* Check your intermediate answers with the answer key as you work.
* When the design example is complete, fill in the blanks in the table on the last page. Nothing is due, but you will use this material for Experiment 2 project.

# Uncertainty analysis in generalized forms

## Error propagation.

Consider a general case in which an experimental result *R* is a function of *n* measured variables *xi*,

(1)

Assuming that the function is continuous with continuous derivatives in the domain of interest, that the measurands *xi* are independent of one another, that the uncertainties in the measurands are independent of one another, and that random uncertainty is expressed in the resultant, then the uncertainty in the result is given by

|  |  |
| --- | --- |
| . | (2) |

The partial derivative terms are *sensitivity coefficients*. The magnitude of a coefficient tells us how sensitive the total uncertainty is to changes in the uncertainty in a measurand.

## RSSC—Root-sum-square contributions.

We often want to compare the contributions of individual measurands and random uncertainty to the total uncertainty in the resultant. We will call these contributions the *root-sum-square-contributions* (RSSCs), defined as the set of values that when squared and summed produce the square of the total uncertainty in the resultant.

|  |  |
| --- | --- |
|  | (3) |

Note that in this form the uncertainty is a *dimensional* value; it has physical units.

## RSSC—percentage form.

To compare individual contributions in percentage form, divide every term in (3) by *R*2.

|  |  |
| --- | --- |
|  | (4) |

The ratio *wR*/*R* is called the *relative uncertainty* in the resultant—usually reported as a percentage, e.g.,



## UMF—Uncertainty Magnification Factor form.

It is useful to manipulate (4) further to obtain relative uncertainties for all measurands. We multiply each term on the right-hand side by (*xi* /*xi*)2 (equals 1, of course) except the random term.

|  |  |
| --- | --- |
|  | (5) |

Now each of the uncertainties of the measurands is in relative form. The coefficients of the relative uncertainties, before squaring, are called *uncertainty magnification factors* (UMFs). UMFs play a role in the *design phase* of an experiment, helping us determine which measurands might require special care.

Comparing (4) and (5), note that the UMF relates the relative uncertainty in a measurand to its RSSC,

|  |  |
| --- | --- |
|  | (6) |

If the relative uncertainty is a percentage, the RSSC is also a percentage.

## UPC—Uncertainty Percentage Contribution form.

The second of the two non-dimensional forms of the uncertainty relationship is in terms of the *uncertainty percentage contribution* (UPCs). To obtain this form, divide all terms on both sides of (5) by the (squared) relative uncertainty term on the left-hand side,

|  |  |
| --- | --- |
|  | (7) |

Each term on the right-hand side indicates the *uncertainty percentage contribution* (UPC) of that measurand to the total squared uncertainty in the resultant. The left-hand side is 1, or 100%.

The UPCs of the measurands are especially useful in the *design stage* of an experiment; they show which measurands have the greatest effect on the uncertainty of the resultant.

## Example 1.

A thermal mass flow-meter (a schematic is given in Figure 1) can be used to measure mass flow rates of clean gases. The data reduction equation is given by

|  |  |
| --- | --- |
| , | (8) |

where  is specific heat and the measurands are heat rate  and temperature  and . The uncertainty in the mass flow rate :

|  |  |
| --- | --- |
| . | (9) |

Sensitivity coefficients (partial derivative terms):

|  |  |
| --- | --- |
| . | (10) |

*UMF form*: Substitute the partial derivative from(10) in (9), then divide every term in (9) by ,

|  |  |
| --- | --- |
| , | (11) |

which simplifies to

|  |  |
| --- | --- |
| . | (12) |

Next multiply the terms on the right-hand side by  as needed to obtain relative uncertainties,

|  |  |
| --- | --- |
| , | (13) |

which is rearranged to obtain the desired UMF form

|  |  |
| --- | --- |
| . | (14) |

*UPC form*: Divide every term in (14) by 

|  |  |
| --- | --- |
|  |  |

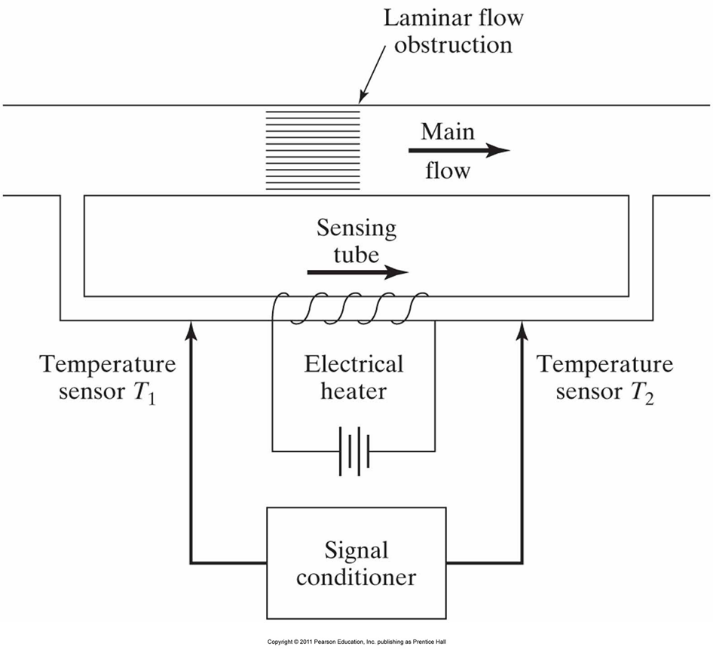


Figure 1 Thermal mass flow rate

## Example 2.

Now suppose we have some estimated values for the mass flow-rate experiment:

|  |  |  |
| --- | --- | --- |
| 0 C | 2 C | 4200 J/kgK |
| 100 C | 2 C | kg/s |
| 5 kW | 3 W | and the estimated random uncertainty in  is 1% |

Determine each UMF. Which UMF(s) have the greatest effect on the uncertainty in the result?

Show that the total relative uncertainty in the resultant is 3%

Determine each UPC and check that they sum to 100%

# Uncertainty budget

|  |  |  |  |  |  |  |  |
| --- | --- | --- | --- | --- | --- | --- | --- |
| Parameter | Representative value | Uncertainty | Basis for uncertainty  (with page no.) | Relative  uncertainty (%) | UMF | RSSC (%) | UPC (%) |
|  |  |  |  |  |  |  |  |
| *T1* |  |  |  |  |  |  |  |
| *T2* |  |  |  |  |  |  |  |
| random |  |  |  |  |  |  |  |
|  | expected value: | — | — | — |  |  |  |

# Using relative uncertainties with inequalities in design

When we study the feasibility of an experiment design, we find that relative uncertainties appear in expressions using inequalities. The following example illustrates how the inequality leads to minimum or maximum allowable values of some measurands.

## Example 4.

Using relative uncertainties with target inequalities.

Suppose we are measuring an angular displacement  in degrees and we have a design constraint,

|  |  |
| --- | --- |
| . | (15) |

We propose to use a sensor with an accuracy of ± 1° and a least significant digit readout of ± 1°.

What limit does the inequality (15) impose on our experiment design?

# Reminder: pressures and temperatures in absolute units

When you encounter relative uncertainty in temperature,

 temperature must be expressed in absolute units, K or R

When you encounter relative uncertainty in pressure,

 pressure must be absolute pressure (not gage pressure)

# Measuring T or p

## Units.

When you encounter expressions involving temperature or pressure *differences*, absolute units are *optional* because *T*1 – *T*2 yields the same value in K and °C and *p*1 – *p*2 yields the same value in absolute and gage pressure units.

## Uncertainties.

Uncertainty analysis involving a *T* or a *p* depends on the exact form of the measurand:

1. *Separate measurements* of *T*1 and *T*2 (or *p*1 – *p*2): You have two sensors, one giving a reading for variable 1 and the other giving a reading for variable 2. Both sensors have systematic uncertainties. To find *T*, you use a data reduction equation, *T* = *T*1 – *T*2. It follows (as with any DRE), the uncertainty in *T* is given by



1. *Direct measurement* of a temperature differential *T* or a pressure differential *p*: Such measurements are made with *differential transducers*, designed to respond to the difference between two input values.

In such cases the measurand is the *difference*, *T* or *p,* and the systematic uncertainty is just

 or  .

# Shortcut for some DREs

A shortcut to find the UMF form of the uncertainty equation is available if the data reduction equation has the form

,

where the *xi* variables are measurands*,* theexponents are constants ( or ), and *K* is a constant.

The UMF form is:



## Example 3.

When can we use the shortcut?

1. *CAN use the shortcut* for the following DRE if current *i* and resistance *R* are the measurands,

.

Thus 

2. *CANNOT use the shortcut* for the following DRE because of the sine function,

 .

Thus 

3. *CANNOT use the shortcut* for the following DRE because of the subtraction of the two pressure measurands,

.

Thus 

4. *CAN use the shortcut* for the following DRE because *p* is measured directly,

.

Thus 

Workshop 8 Design Example

Design an experiment to determine the modulus of elasticity

A cantilever beam experiment is proposed to determine the modulus of elasticity *E* of the beam material. The beam bends due to an applied force *P* located a distance *x* from the center of the strain gages. The gages are configured to respond to bending only. For each value of *P*, a strain reading  is recorded.

cantileverbeam

Fig. 1: A cantilever beam system for determining modulus of elasticity.

The measurands are *P*, *x*, , *b*, and *h* and the data reduction equation is given by

|  |  |
| --- | --- |
|  | (16) |

|  |
| --- |
| ***The design task.*** Determine ranges of values for *L*, *b*, *h*, *x*, **, and *P* to determine the modulus of elasticity *E* with an uncertainty no greater than ±5%. Determine the feasibility of the design, select instrumentation, and identify measurements that require special care. Determine which of the design variables should be assigned set-points. Select and justify the set-point values. Determine which of the design variables should be assigned a range of values in the test sequence. Select and justify the ranges. Develop an uncertainty budget. |

# Setting targets for uncertainties in each measurand

Recall that the data reduction equation is given by

|  |  |
| --- | --- |
|  | (16) |

Develop, in UMF form, the expression for the uncertainty in *E*.

|  |  |
| --- | --- |
|  | (17) |

Since the height *h* has a *UMF* of 2 and each of the other measurands has a *UMF* of 1, the relative uncertainty in *h* has a greater effect than the other measurands on the uncertainty in E. The relative uncertainty in *h* can be controlled by specifying a small machining tolerance for *h*.

***Explore the uncertainty equation***.

As a starting point, assume all the relative uncertainty terms in (17), including relative random uncertainty, are equal to each other. Find the value of this relative uncertainty that will produce a 5% uncertainty in the experimental result.

*Ans.:* ±1.667%

*Comment:* To give ourselves some leeway later (making room for unforeseen uncertainty sources and additional random uncertainty in the data-analysis phase),

let’s use ±1.5%

as our *target relative uncertainty* for each measured variable.

# Feasibility.

We will investigate experiment feasibility one measured variable at a time. Use US customary units throughout the workshop.

 *Uncertainty of applied force P*. Goal: *wP*/*P* ≤ 1.5%.

*Method*: Assume that forces *P* are applied to the beam using only the weights found on the cart. (If this assumption proves to be a poor one, we will have to acquire a new set of weights or propose an alternative conceptual design for force application.)

Using both sets of weights combined, what is the range of *P*? (1.0 kg *weighs* 2.207 lbf.)

available: \_\_\_\_\_\_\_\_\_\_ ≤ *P* ≤ \_\_\_\_\_\_\_\_\_\_ lbf

|  |  |
| --- | --- |
| Assume we use the platform balance shown here to measure *P*. The transducer specs are:  Capacity: 32 kg  Readability acc: 1 g  Repeatability: 0.6 g  Linearity: ± 1 g | Shimadzu BW-32K Platform Balance |

Determine the range of forces *P* that satisfy the relative-uncertainty design goal. Report *P* to one decimal place only, rounding up for the minimum and rounding down for the maximum

Design range: \_\_\_\_\_\_\_\_\_\_ ≤ *P* ≤ \_\_\_\_\_\_\_\_\_\_ lbf

|  |  |
| --- | --- |
| *Uncertainty of distance x*. Goal: *wx*/*x* ≤ 1.5%. | cantileverbeam  Beam side-view |

*Method*: Reviewing materials in stock, we find that we have bar stock up to 24 inches in length. A tape measure is on the cart. (No information on accuracy? Use the readability uncertainty as an estimate.)

*Tape measure.* Determine the smallest divisions on the tape measure. Some tapes have smaller resolution in the first 12 inches than in regions over 12 inches.

Determine the range of distances that satisfies the relative-uncertainty design goal. Again…use the inequality! Round up to one decimal place. Show all your work.

Tape design range: \_\_\_\_\_\_\_\_\_\_ ≤ *x* ≤ \_\_\_\_\_\_\_\_\_\_ in

|  |  |
| --- | --- |
| *Uncertainty of bar thickness h*. Goal: *wh*/*h* ≤ 1.5%. | cantileverbeam  Beam end-view |

*Method*: From the UMF uncertainty equation (17), we know a small uncertainty in *h* is desirable. We can meet this requirement by having the bar machined to a small tolerance. The machining *tolerance* is our uncertainty—bar thickness *h* does not have to be measured.

Suppose the part is easily machined with a tolerance of ±0.005 in. Determine the minimum value for *h*. Use the inequality! Since this is a minimum, round up to the nearest 0.001 inch. Show your work.

*Ans*: hmin = 0.334 in

Reviewing our supplies, we find that ½-in thick bar-stock is readily available. Using this metal stock, the range of possible thicknesses we might specify is

|  |  |
| --- | --- |
| in. |  |

We want the machinist to remove *some* material so we don’t have to rely on the manufactured tolerances. Suppose we agree that

|  |  |
| --- | --- |
| in. |  |

is the specification we send to the machine shop. Show that this specification meets our target relative uncertainty goal for *h*.

*Ans*:  \_\_\_\_\_\_\_\_\_\_\_\_\_%

*Conversation with our machinists:*

* Machining a part generally requires more time & money as tolerances gets smaller. If we could obtain some stock metal with *h* ≥ 0.667 in, we could get by with a ±0.010 tolerance.
* If we need it, we can get a tolerance of ±0.001 in using the grinder if we limit the length *L* ≤ 18 in. Let’s keep *x*max = 24 in for now, but keep this possibility in mind later if we need to reduce *wh*.

|  |  |
| --- | --- |
| *Uncertainty of bar width b*. Goal: *wb*/*b* ≤ 1.5%. | cantileverbeam  Beam end-view |

*Method*: The bars we have in stock come in three widths: ½ in, ¾ in, and 1 in. Our machinists say that using our mill, the *best* tolerance we can obtain for *b* is ±0.005 but they prefer we go with ±0.010 if we can.

Using the ±0.010 in tolerance, do any of the bar widths meet our uncertainty target?

Again, we want the machinist to remove *some* material so we don’t have to rely on the manufactured tolerances. Suppose we agree to use the 1 inch stock and send the following specification to the shop:

|  |  |
| --- | --- |
| in. |  |

Show that this specification meets our target relative uncertainty goal for *b*.

*Ans*:  \_\_\_\_\_\_\_\_\_\_\_\_\_%

|  |  |  |
| --- | --- | --- |
|  | *Uncertainty of strain measurement *.  Goal: *w*/** ≤ 1.5%. | cantileverbeam  Beam side-view |

*Method*: The blue box on the cart is a strain indicator (by Vishay). The strain indicator accuracy includes a “count”, defined as the least significant digit in the readout, which for this readout is 1  (one microstrain =  m/m).

* the uncertainty due to accuracy, 
* the uncertainty due to readability, = 0.5

Determine the smallest strain reading consistent with relative-uncertainty design goal. Show all your work.

*Ans:* **min = 217  .

# Reality checks.

All combinations of *P* and *x* should produce a bending moment that results in a strain greater than **min without exceeding the yield strength of the material. To investigate, rearrange the DRE to place the bending moment on the left-hand side of the equation,

|  |  |
| --- | --- |
|  | (18) |

*Given*

*h* = 0.400 in, *b* = 0.900 in, and **min = 217 .

Assume the bar is aluminum, with *E* ≈×6 psi. (We use an estimated value of the *resultant E* to investigate this condition.)

*Find*

The limit on *Px* imposed by  ≥ min.

*Answer*: *Px* *≥* \_\_\_\_\_\_\_\_\_\_\_\_\_\_ in-lb.

|  |  |
| --- | --- |
| ***Design space***.  On the figure,   * Sketch the limits: 3 ≤ *x* ≤ 24 in * Sketch the limits: 0.3 ≤ *P* ≤ 9.2 lb * Identify the *design space*, within which any *Px* combination satisfies the design goal.   ***Reality check on bending stress***.  The material yield strength imposes a maximum for the bending moment. If we substitute stress  for *E* in (18), we obtain    *Given*  Aluminum yield strength of 40 ksi.  Factor of safety (FOS) of 4.  *Find*  Limit on *Px* due to yield strength and FOS.  Sketch it approximately on the figure. |  |

*Answer*: *Px* *≤* \_\_\_\_\_\_\_\_\_\_\_\_\_\_ in-lb

***Repeat for a steel bar.***

*Given*

*h* = 0.400 in, *b* = 0.900 in, and **min = 217 .

for steel, *E* ≈×6 psi.



*Find*

The limit on *Px* imposed by  ≥ min.

*Answer*: *Px* *≥* \_\_\_\_\_\_\_\_\_\_\_\_\_\_ in-lb.

|  |  |
| --- | --- |
| ***Design space***  On the figure, the limits are shown for  3 ≤ *x* ≤ 24 in  0.3 ≤ *P* ≤ 9.2 lb  Identify the *design space* for steel, within which any *Px* combination satisfies the design goal.  ***Reality check on bending stress***.  *Given*  Steel yield strength of 60 ksi.  Factor of safety (FOS) of 4.      *Find*  Limit on *Px* due to yield strength and FOS.  Sketch it approximately on the figure. |  |

*Answer*: *Px* *≤* \_\_\_\_\_\_\_\_\_\_\_\_\_\_ in-lb.

To increase the size of the design space but still keep the bar to 24 in or less, what constraint can we change and how?

|  |  |
| --- | --- |
| T-316-LABB Measurement Systems  *Workshop 7 memo* |  |

Assume the beam material is steel. Based on the feasibility analysis, we select the following values and ranges for our measurands:

6.5 ≤ *P* ≤ 9.2 lb

*x* = 24 in,

*L* = 30 in,

*h* = 0.400 ± 0.005 in*,*

*b* = 0.900 ± 0.010 in

** > 217 

Using these design values, create an “Uncertainty Budget for Design” table like the one shown on the next page.

* In the experiment, we use the full range of weights for *P*. For the uncertainty budget table, we select a representative constant. Let’s agree to use *P* = 7 lb as our representative value in the budget. Then all values in the budget (, for example) that depend on *P* are computed using 7 lb.
* Complete your table.
* Rearrange the rows of the table in order of decreasing magnitude of the UPC, from largest UPC in the top row to the smallest UPC in the last row.
* Check that the root-sum-square of the RSSCs is the resultant relative uncertainty .
* Check that the sum of the UPCs is 100%.
* Round relative uncertainties and RSSCs to the nearest 0.01%. Anything smaller is reported as “< 0.01%”.
* Round UPCs to the nearest 1%. Anything smaller is reported as “< 1%”.
* If the final uncertainty is less than the original target (±5%), then the feasibility of the experiment design is supported.

# Uncertainty budget for design

|  |  |  |  |  |  |  |  |
| --- | --- | --- | --- | --- | --- | --- | --- |
| Parameter | Representative value | Uncertainty | Basis for uncertainty  (with page no.) | Relative  uncertainty (%) | UMF | RSSC (%) | UPC (%) |
| *Erandom* | — | — | Budget allowance  (p. 12) | 1.50 | 1 | 1.50 |  |
| *P* | 7 lb | ± 0.0034 lb | Scale accuracy & readability  (p. 14) | 0.05 | 1 | 0.05 |  |
| *x* |  |  |  |  |  |  |  |
| *h* |  |  |  |  |  |  |  |
| *b* |  |  |  |  |  |  |  |
|  |  |  |  |  |  |  |  |
| *E* | expected value: | — | — | — |  |  |  |

|  |  |
| --- | --- |
| cantileverbeam  Fig. 1: A cantilever beam system for determining modulus of elasticity. | Data reduction equation for design    Uncertainty equation for design |

# Uncertainty budget for design – arranged in order of percent contribution

|  |  |  |  |  |  |  |  |
| --- | --- | --- | --- | --- | --- | --- | --- |
| Parameter | Representative value | Uncertainty | Basis for uncertainty  (with page no.) | Relative  uncertainty (%) | UMF | RSSC (%) | UPC (%) |
|  |  |  |  |  |  |  |  |
|  |  |  |  |  |  |  |  |
|  |  |  |  |  |  |  |  |
|  |  |  |  |  |  |  |  |
|  |  |  |  |  |  |  |  |
|  |  |  |  |  |  |  |  |
| *E* | expected value: | — | — | — |  |  |  |